A Formula for the Partial Fractions Decomposition of $x^n/(x-a)^k$

Justin Skycak

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Introduction

While conducting numerical experiments with partial fractions decomposition, I observed the following pattern:

$$\frac{x^n}{(x-a)^k} = \sum_{i=0}^{n-k} \binom{n-1-i}{k-1} a^{n-k-i} x^i + \sum_{i=\max(k-n,1)}^{k} \frac{\binom{n}{k-i} a^{n-k+i}}{(x-a)^i}$$

for $n, k \in \mathbb{N}$. The binomial coefficients are taken to be 0 where they are otherwise undefined.

A proof is provided below. Double induction is used, abbreviating the above proposition as $P(n,k)$.

First Base Case

We prove $P(1,1)$ as the basis for the first induction:

$$\begin{aligned}\frac{x}{x-a} &= \sum_{i=0}^{0}\binom{-i}{0}a^{-i}x^{i} + \sum_{i=\max(0,1)}^{1}\frac{\binom{1}{1-i}a^{i}}{(x-a)^{i}} \\ &= 1 + \frac{a}{x-a} \\ &= \frac{x}{x-a}\end{aligned}$$

First Inductive Step

Assume $P(n,1)$ for $n \in \mathbb{N}$. We will show that $P(n+1,1)$ follows.

First, we take the following notation in the theorem to be proved:

$$\frac{x^{n}}{(x-a)^{k}} = p(m,k) + f(m,k)$$

where

- $p(n,k) = \sum_{i=0}^{n-k}\binom{n-1-i}{k-1}a^{n-k-i}x^{i}$ is the polynomial part, and
- $f(n,k) = \sum_{i=\max(1,n-k)}^{k}\frac{\binom{n}{k-i}a^{n-k+i}}{(x-a)^{i}}$ is the fractional part.

Note the following:

$$
\begin{aligned}
p(n,1) &= \sum_{i=0}^{n-1} \binom{n-1-i}{0} a^{n-1-i} x^i \\
&= \sum_{i=0}^{n-1} a^{n-1-i} x^i \\
f(n,1) &= \sum_{i=\max(1-n,1)}^{n} \frac{\binom{n}{1-i} a^{n-1+i}}{(x-a)^i} \\
&= \sum_{1}^{n} \frac{\binom{n}{1-i} a^{n-1+i}}{(x-a)^i} \\
&= \frac{a^n}{x-a}
\end{aligned}
$$

Additionally,

$$
\begin{aligned}
p(n+1,1) &= \sum_{i=0}^{n} \binom{n-i}{0} a^{n-i} x^i \\
&= \sum_{i=0}^{n} a^{n-i} x^i \\
&= a^n + \sum_{i=1}^{n} a^{n-i} x^i \\
&= a^n + \sum_{i=0}^{n-1} a^{n-(i+1)} x^{i+1} \\
&= a^n + x \sum_{i=0}^{n-1} a^{n-1-i} \\
&= a^n + xp(n,1)
\end{aligned}
$$

and

$$
\begin{aligned}
f(n+1,1) &= \sum_{i=\max(-n,1)}^{n+1} \frac{\binom{n+1}{1-i} a^{n+i}}{(x-a)^i} \\
&= \sum_{1}^{n+1} \frac{\binom{n+1}{1-i} a^{n+i}}{(x-a)^i} \\
&= \frac{a^{n+1}}{x-a} \\
&= af(n,1).
\end{aligned}
$$

Finally, using $P(n,1)$, we have

$$
\begin{aligned}
\frac{x^{n+1}}{x-a} &= x\left(\frac{x^n}{x-a}\right) \\
&= x\left(p(n,1)+f(n,1)\right) \\
&= x\left(p(n,1)+\frac{a^n}{x-a}\right) \\
&= xp(n,1)+a^n\left(\frac{x}{x-a}\right) \\
&= xp(n,1)+a^n\left(1+\frac{a}{x-a}\right) \\
&= xp(n,1)+a^n+\frac{a^{n+1}}{x-a} \\
&= p(n+1,1)+f(n+1,1),
\end{aligned}
$$

which proves $P(n+1,1)$ as desired.

First Inductive Conclusion (Second Base Case)

We have proven $P(1,1)$ and shown that $P(n,1) \implies P(n+1,1)$ for $n \in \mathbb{N}$. Therefore, $P(n,1)$ for all $n \in \mathbb{N}$.

Second Inductive Step

Assume $P(n,k)$ for $n, k \in \mathbb{N}$. We will show that $P(n,k+1)$ follows.

Using $P(n,k)$, we have

$$\begin{aligned}
\frac{x^n}{(x-a)^{k+1}} &= \frac{1}{x-a}\left(\frac{x^n}{(x-a)^k}\right) \\
&= \frac{1}{x-a}\left(p(n,k) + f(n,k)\right) \\
&= \frac{p(n,k)}{x-a} + \frac{f(n,k)}{x-a}.
\end{aligned}$$

We will now re-express each term in the sum above.

Using Pascal's identity in the form

$$\binom{n-1-i}{k-1} = \binom{n-i}{k} - \binom{n-1-i}{k},$$

we have

$$
\begin{aligned}
\frac{p(n,k)}{x-a} &= \sum_{i=0}^{n-k} \binom{n-1-i}{k-1} a^{n-k-i} x^i \\
&= \sum_{i=0}^{n-k} \binom{n-i}{k} a^{n-k-i} x^i - \sum_{i=0}^{n-k} \binom{n-1-i}{k} a^{n-k-i} x^i \\
&= \sum_{i=-1}^{n-k-1} \binom{n-i-1}{k} a^{n-k-1-i} x^{i+1} - a \sum_{i=0}^{n-k} \binom{n-1-i}{k} a^{n-k-1-i} x^i \\
&= \binom{n}{k} a^{n-k} + \sum_{i=0}^{n-k-1} \binom{n-i-1}{k} a^{n-k-1-i} x^{i+1} \\
&\quad - a \left[\binom{k-1}{k} a^{-1} x^{n-k} + \sum_{i=0}^{n-k-1} \binom{n-1-i}{k} a^{n-k-1-i} x^i \right] \\
&= \binom{n}{k} a^{n-k} + x p(n,k+1) - a\left[0 + p(n,k+1)\right] \\
&= \binom{n}{k} a^{n-k} + (x-a) p(n,k+1).
\end{aligned}
$$

Simplifying the second term in the sum, we have

$$
\begin{aligned}
\frac{f(n,k)}{x-a} &= \sum_{i=\max(k-n,1)}^{k} \frac{\binom{n}{k-i} a^{n-k+i}}{(x-a)^i} \\
&= \sum_{i=1}^{k} \frac{\binom{n}{k-i} a^{n-k+i}}{(x-a)^i} \text{ (introducing terms equal to 0)} \\
&= \sum_{i=2}^{k+1} \frac{\binom{n}{k+1-i} a^{n-k-1+i}}{(x-a)^i} \\
&= \sum_{i=1}^{k+1} \frac{\binom{n}{k+1-i} a^{n-k-1+i}}{(x-a)^i} - \frac{\binom{n}{k} a^{n-k}}{x-a} \\
&= f(n,k+1) - \frac{\binom{n}{k} a^{n-k}}{x-a} \text{ (removing terms equal to 0).}
\end{aligned}
$$

Finally, we substitute into our original sum and reach

$$\begin{aligned}\frac{x^n}{(x-a)^{k+1}} &= \frac{p(n,k)}{x-a} + \frac{f(n,k)}{x-a} \\ &= \frac{\binom{n}{k}a^{n-k} + (x-a)p(n,k+1)}{x-a} + f(n,k+1) - \frac{\binom{n}{k}a^{n-k}}{x-a} \\ &= p(n,k+1) + f(n,k+1),\end{aligned}$$

as desired.

Second Inductive Conclusion

We have proven $P(n,1)$ and shown that $P(n,k) \implies P(n,k+1)$ for $n,k \in \mathbb{N}$. Therefore, $P(n,k)$ for all $n,k \in \mathbb{N}$.